Building Research Establishment Digest

CI/SfB 1976 (F47) | DIGEST 229

September 1979

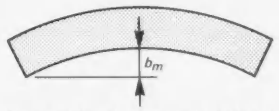
Estimation of thermal and moisture movements and stresses: Part 3

Estimation of deformations and associated forces and stresses

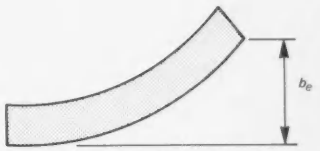
This part of the digest provides expressions for the assessment of thermal deformations, forces and stresses in those cases that are amenable to reasonably simple analysis. All the expressions are to some extent approximate, either with respect to the geometry of curvature, or the estimation of forces and stresses, or both. Also, they deal with curvature in only one plane, and may therefore be inadequate for the analysis of members whose breadth is of the same order as their length. Some guidance on two-dimensional stress analysis is given in Ref (1). However, the expressions will generally enable these effects to be assessed with sufficient accuracy for the purposes of building design, bearing in mind the underlying uncertainties about the precise properties of many materials and the conditions to which they are subject.

Cases (a), (b) and (c) in Figure 1 are dealt with individually, and expressions are included both for the estimation of the deformations occurring in the absence of external restraint and for the forces and stresses induced if full lateral restraint is provided. While in practice the magnitude of the deformations or restraining forces might be required at a number of different points, the following two general cases are considered, Figs 3 and 4.

As the restraining forces are determined using transpositions of standard expressions for structural deflections, expressions giving the forces involved in other systems of restraint, eg uniformly distributed forces, can be easily derived.

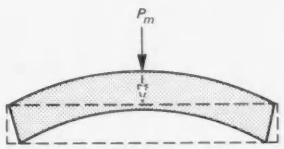


Bow at midlength, b_m, assuming both ends free to rotate

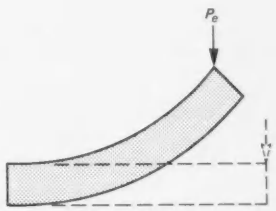


Bow at end, b_e, assuming other end fixed

Fig 3 Deformations



Restoring force at midlength, P_m



Restoring force at end, P_e

Fig 4 Forces

Prepared at Building Research Station, Garston, Watford, WD2 7JR

Technical enquiries arising from this Digest should be directed to Building Research Advisory Service at the above address.

All the expressions may be adapted for assessing the effects of changes in moisture content by replacing the product 'αt' by a value for strain, calculated as a proportion:

$$\text{Strain} = \frac{\text{change in length}}{\text{overall length}} = \frac{\text{Factor (\%)}}{100}$$

Values for the elastic and geometric properties of sections are needed in a number of cases, and some expressions for these are given first.

Elastic and geometric properties of sections

When the analysis involves bending moments, it is necessary to determine the stiffness of the member under consideration. Stiffness is defined as the product of the modulus of elasticity E and the second moment of area of the section, I. For homogeneous materials, the value of I can be determined from the cross-sectional dimensions of the member. For composite and laminated members, a value for EI, the 'equivalent stiffness', must be calculated from the appropriate expression. In all cases, values for E will be needed (see Table 1, Column 5).

Homogeneous material of rectangular section

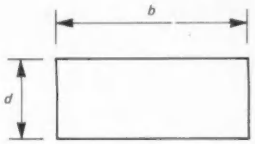


Fig 5

$$I = \frac{bd^3}{12}$$

Expressions for many other section shapes can be found in standard references on structural analysis.

Reinforced concrete (simplified expressions assuming single reinforcement and neutral axis at a depth of $\frac{d_1}{2}$)

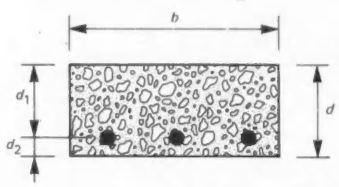


Fig 6

E_c = modulus of elasticity, concrete

E_{st} = modulus of elasticity, steel

A_{st} = total cross-sectional area, steel

Where thermal bending effects act in the same direction as the bending load for which the section is designed:

$$\text{Equivalent } EI = \frac{bd^3 E_c}{12} + \frac{(E_{st} - E_c) A_{st} d_1^2}{4}$$

Where thermal bending effects act in the opposite direction to the bending load for which the section is designed:

$$\text{Equivalent } EI = \frac{bd^3 E_c}{12} + \frac{(E_{st} - E_c) A_{st} d_2^2}{4}$$

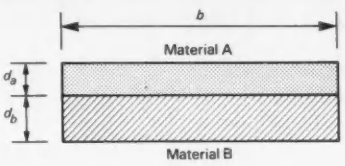


Fig 7

Moduli of elasticity of materials = E_a and E_b

Let $E_a d_a = Q_a$ and $E_b d_b = Q_b$, then

Equivalent $EI = \frac{b}{12}\left\{Q_a d_a^2 + Q_b d_b^2 \quad \frac{3 Q_a Q_b (d_a + d_b)^2}{Q_a + Q_b}\right\}$

Thermally-homogeneous material subject to different environments on opposite faces - as in Fig 1(a)

Conditions: The material is undeformed when at a uniform temperature of T_1. It deforms as illustrated when the surface temperatures are T_1 and T_2.

Assumptions: The temperature gradient across the thickness of the material is uniform.
Shear strain is neglected; this means that the expressions given below will be less accurate for some materials of low elastic modulus and correspondingly low shear modulus.

Data required: L, d and b (breadth of section)
α, T_1 and T_2 (or data on relative strain to be substituted for αt)
E
I calculated from b and d (or as appropriate for composite materials)
t calculated as $T_2 - T_1$

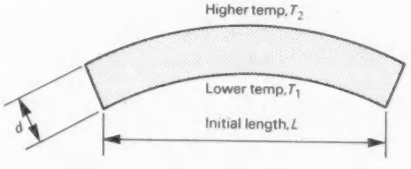


Fig 8 Deformed shape of material

If no external restraints are imposed, and the material is free to deform:

Bow at midlength, $b_m = \frac{L^2}{8d} \times$ difference in strain between surfaces

$$= \frac{\alpha t L^2}{8d}$$

Bow at end, $b_e = 4b_m = \frac{\alpha t L^2}{2d}$

Stress in material = 0

If material is restrained from taking up a deformed shape:

(a) by a point force at midlength, P_m

$$P_m = \frac{48 b_m EI}{L^3} = \frac{6 \alpha t EI}{dL}$$

(b) by a point force at end, P_e

$$P_e = 0.25 P_m = \frac{3 b_e EI}{L^3} = \frac{1.5 \alpha t EI}{dL}$$

Bending moment, $M = \frac{P_m L}{4} = P_e L$

Maximum bending stress $= \frac{Md}{2I}$

Note: for materials, such as reinforced concrete, this represents the extreme fibre stress in the matrix; where necessary, I should be determined for this purpose from $I = \frac{\text{Equiv } EI}{E_{matrix}}$

Example of magnitude of effects

A reinforced concrete slab subject to thermal bending in the opposite direction to its bending under load (eg, a singly-reinforced roof slab 'hogging' due to solar heating of its upper surface). Length 3.5 m, width 2.0 m, thickness 100 mm; $t = 30°C$; $\alpha = 12 \times 10^{-6}/°C$; $E_c = 30 kN/mm^2$; $E_{st} = 210 kN/mm^2$; $A_{st} = 1200 mm^2$; $d_1 = 75$ mm; $d_2 = 25$ mm. Calculated values:

If free to deform: $b_m = 5.5$ mm

If restrained by a point force at midlength: $P_m = 31$ kN

Max bending stress $= 8.2 N/mm^2$

Two joined or bonded materials subject to a uniform change in environment - as in Fig 1 (b)

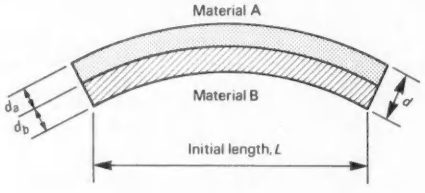


Fig 9 Deformed shape of materials

Conditions: The joined materials are undeformed when at a uniform temperature of T_1. They deform as illustrated when the temperature of both materials changes to T_2.

Assumptions: Shear strain is neglected

The bond between the materials does not strain, ie there is no shear lag at the interface

Data required: L, d, d_a, d_b, and b (breadth of section)
α_a, α_b, T_1 and T_2 (or data on strains to be substituted for $\alpha_a t$ and $\alpha_b t$)
I_a calculated from b and d_a
I_b calculated from b and d_b
Equivalent EI of the combined section, calculated as shown in *Elastic and geometric properties of sections*
t calculated as $T_2 - T_1$

Calculation of curvature

If the radius of curvature = ρ, then

$$\frac{1}{\rho} = \frac{(\alpha_b - \alpha_a)t}{\frac{d}{2} + \frac{2(E_a I_a + E_b I_b)}{bd}\left(\frac{1}{E_a d_a} + \frac{1}{E_b d_b}\right)}$$

If no external restraints are imposed, and the member is free to deform:

Bow at midlength, $b_m = \frac{L^2}{8}\frac{1}{\rho}$

Bow at end, $b_e = 4b_m = \frac{L^2}{2}\cdot\frac{1}{\rho}$

If the member is restrained from taking up the deformed shape:

(a) by a point force at midlength, P_m

$$P_m = \frac{48\, b_m EI}{L^3}$$

(b) by a point force at end, P_e

$$P_e = 0.25P_m = \frac{3b_e EI}{L^3}$$

where EI is the equivalent property of the combined section.

The maximum stress in each material will occur at the interface. As the temperature rises, this will take the form of compression in the material with the greater coefficient of expansion (Material A in Fig 9) and tension in the material with the lesser coefficient of expansion (Material B). These stresses reduce uniformly with distance from the interface, and in many combinations of material thickness and moduli of elasticity there will be stress reversal towards the outer layers of each material.

The maximum (interface) stresses can be calculated as follows:

Max stress in Material A $= \frac{1}{\rho}\left|\frac{2}{bdd_a}\left(E_a I_a + E_b I_b\right) + \frac{d_a E_a}{2}\right|$

Max stress in Material B $= \frac{1}{\rho}\left|\frac{2}{bdd_b}\left(E_a I_a + E_b I_b\right) + \frac{d_b E_b}{2}\right|$

For source of expressions see Ref (2)

Example of magnitude of effects

A 2 mm thick melamine laminate bonded to a 15 mm plywood sheet, the plywood being subject to moisture movement; length 1.5 m; width 0.5 m; Laminate E = 7 kN/mm^2 ; plywood moisture movement 0.15% (half the overall cross-grain movement, taken as 0.3%); E = 10 kN/mm^2 .

If free to deform: b_m = 12 mm

If restrained by a point force at midlength:

P_m = 326 N
Max stress in laminate = 7.5 N/mm^2
Max stress in plywood = 4.2 N/mm^2

Homogeneous material, eccentrically restrained

Conditions: The material is undeformed when at a uniform temperature of T_1. It deforms as illustrated in Fig 10 when its temperature rises to T_2 .

Assumptions: The restraining construction permits no axial expansion of the material, but provides no restraint to rotation of the ends of the member.
The member has little flexural stiffness (assumption for the assessment of deformation only).
The member is not subject to external loads.

Data required: L, d (thickness of section) and b (breadth of section)
α , T_1 and T_2 (or data on strain to be substituted for αt)
E

Restraint may be imposed by enclosing construction (Fig 10) or by fixings which are unable to accommodate the linear change of size of a material. Where the member or material is thin and flexible, its loadbearing capacity in compression will be low, and restrained expansion will cause it readily to assume a bowed profile. Its behaviour will be analogous to that of a buckled strut although, unlike struts carrying external loads, this will not cause instability since the force in the member will be partially relieved as a consequence of bowing.

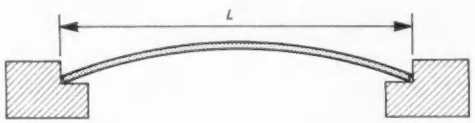


Fig 10 Deformed shape of material

Full analysis of this condition, taking account of the bending resistance of the member, is beyond the scope of this digest. However, a useful guide to the extent of lateral deformation of flexible members can be obtained by ignoring bending resistance and determining the curvature produced if the unrestrained, expanded length is taken as the arc of the curve, with the original length forming the chord. The lateral force required to restrain the bowing can be derived from this geometrical relationship, and the same method can also be used to determine the restraining forces for stiffer members, even though their potential deformations will not be accurately represented.

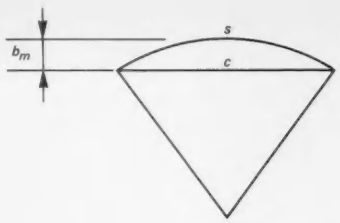


Fig 11 Chord length and arc length

Chord length, c, is taken as the initial length of the member (or its length when full linear restraint is applied, if different); arc length, s, is taken as the unrestrained, expanded length of the member.

Strain in member = $\frac{s-c}{c}$ (= αt, with t appropriately determined)

If no lateral restraint is provided, Fig 12 can be used to determine the amount of bow corresponding to the strain, assuming zero stiffness in the member. Values of the midlength bow, b_m, are given as a proportion of the chord length, c, and use of the graph is illustrated by an example:

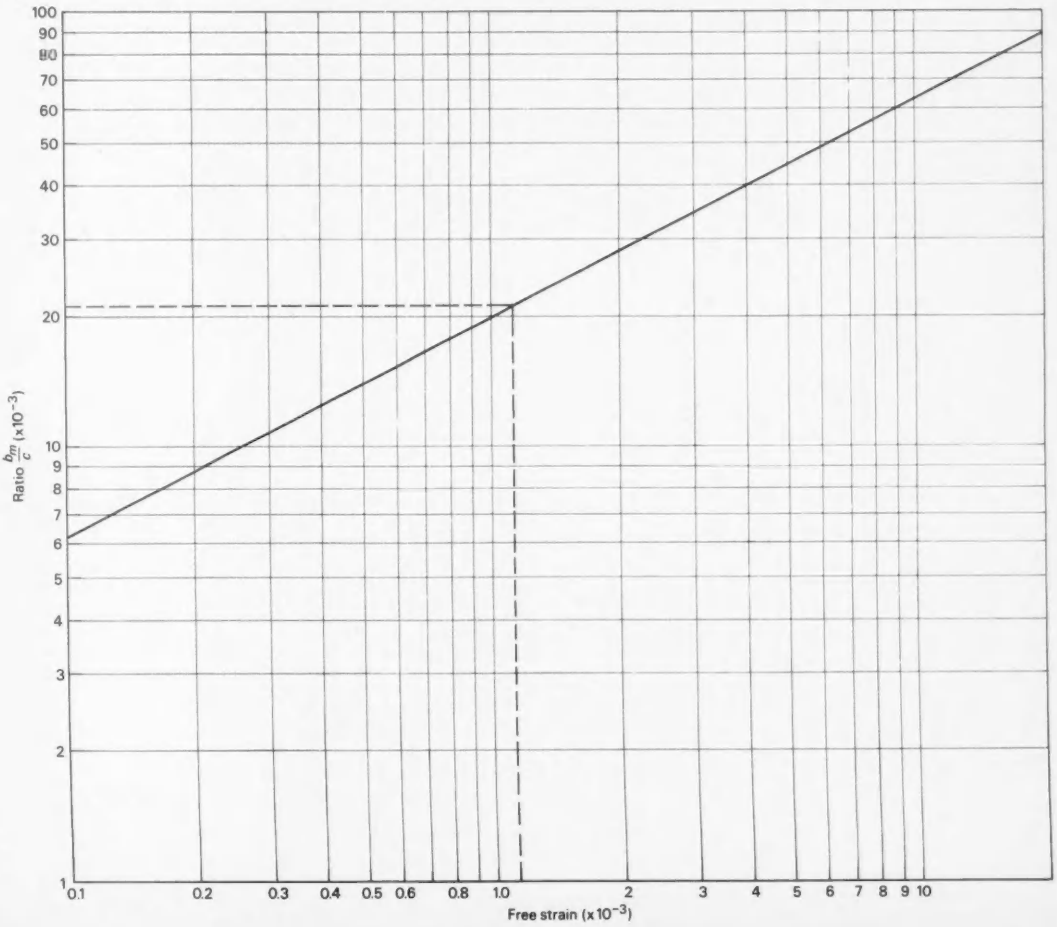
A thin member 2 m long is subject to a strain of 0.00125 (given, for example, by $\alpha = 25 \times 10^{-6}$/°C and $t = 50$°C). As shown by the dotted line on the graph,

$$\frac{b_m}{c} = 0.021$$

Bow, $b_m = 0.021 \times 2000 = 42$ mm

Since bending resistance is ignored, the actual value of b_m will be slightly less than the calculated figure, even for the most flexible member. As noted, for stiffer members the influence of bending resistance becomes dominant and b_m will be significantly less than the 'zero-stiffness' value.

Fig 12 Relationship of bow to arc length and chord length



If lateral restraint is provided, the restraining force will be at a maximum just as bowing commences. At this point bending resistance will be virtually zero. If partial bowing is permitted, the necessary restraining force will reduce, reaching zero when the member assumes its unrestrained profile. The principle of strain energy can be used to equate the work done in forcing the member from its unrestrained to its fully restrained profile with the equivalent work done in restraining axial expansion. Both work totals are based on a force increasing linearly from zero to maximum over the distance in question.

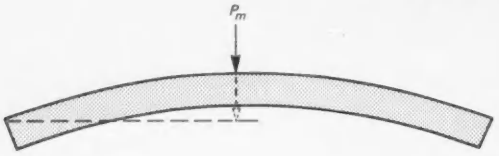


Fig 13 P_m ranges from zero to maximum over distance b_m

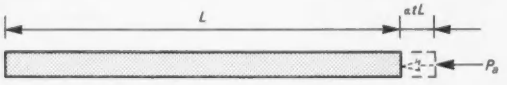


Fig 14 P_a ranges from zero to maximum over distance αtL

Thus, by strain energy equivalence, the maximum value of a central lateral restraining force, P_m, will be:

$$\frac{P_m}{2} \times b_m = \frac{P_a}{2} \times \alpha tL$$

$$\therefore P_m = \frac{P_a\, \alpha tL}{b_m} = \frac{EbdL(\alpha t)^2}{b_m}$$

Stress in material = αtE (as for axial restraint)

The value for unrestrained linear expansion, αtL, may be obtained using Fig 2 and that of the 'zero-stiffness' bow, b_m, as described above, using Fig 12.

These expressions are valid even when the stiffness of the member causes the actual bow to be substantially less than b_m. In such a case the lateral restraining force moves through a smaller distance, the balance of the work being provided by the forces of bending resistance. However, the maximum value of P_m remains the same, since bending resistance will have effectively vanished in the near-straight condition. Restraint by a single point force may not be a valid assumption for very flexible members, which may tend to bow over shorter spans.

Example of magnitude of effect

A plane GRP panel, fully restrained against axial movement; length 1.2m, width 0.6m, thickness 8 mm; $t = 40°C$; $\alpha = 25 \times 10^{-6}/°C$; $E = 5\ kN/mm^2$. Calculated values:

If free to deform: b_m = 24 mm
If restrained by a point force at midlength: P_m = 1.23 kN
Stress in material = 5 N/mm^2

References

(1) Formulas for stress and strain; RJ Roark and WC Young; Fifth edition; McGraw-Hill Book Co. 1975.

(2) Analysis of bi-metal thermostats; S Timoshenko; J Optical Soc Amer, Vol 11, pp 233-255, 1925

Department of the Environment **Building Research Establishment**

© *Crown Copyright 1979*

Published by Her Majesty's Stationery Office and printed litho in Wales by The Five Arches Press, Tenby, Dyfed.
Single copies 14p each. Annual subscription inclusive of postage £2.52 for 12 monthly issues and new editions of earlier Digests.
Quantity rate (excluding postage) for bulk supply of any one Digest: 50 for £5.60.

Dmd. 596159 ISBN 0 11 725038 4